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OPTIMUM STRATEGY WHEN CORRECTING THE TRAJECTORY OF SPACECRAFT

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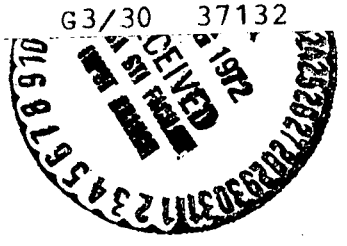

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OPTIMUM STRATEGY WHEN CORRECTING THE TRAJECTORY OF SPACECRAFT

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ABSTRACT. An optimum strategy is compiled for the pulse correction of a spacecraft moving close to the rated trajectory.

This paper is devoted to compiling an optimum strategy for the pulse correction of spacecraft moving close to the rated trajectory. /357*

The correction problem consists of the following. Because of injection errors, the actual spacecraft trajectory deviates from the rated trajectory, as a result of which the final parameters of the actual trajectory differ from the final parameters of the rated trajectory. Having available a system of trajectory measurements giving information as to the actual motion of the spacecraft, it is necessary to select the control method which would enable the mismatch of the indicated parameters to be reduced. A specific fuel consumption is associated with each control method. That control method is preferred which requires a minimum fuel reserve to satisfy the conditions at the final point with a given precision or, what amounts to the same thing, insures maximum precision of fulfilling the final conditions with a given fuel reserve.

The problem of pulse correction has attracted the attention of many researchers [1-5]. In [1-2] two component parts of the correction problem were studied separately: planning the actual trajectory and the correction itself. It was assumed that the correction method consists of two steps:

*Numbers in the margin indicate pagination in the original foreign text.

determining the motion trajectory and carrying out the correction. This method does not use all the possibilities for optimization of the control process. With such a method the correction is begun only after the actual trajectory has become known with a given precision. For a fixed trajectory measurement system, the time necessary for obtaining the trajectory with a given precision may not be less than a certain time which depends on the selected measurement system. After the lapse of the time required for determining the trajectory, the effectiveness of the correcting pulse, generally speaking, is less than the effectiveness of the pulse in the section where the trajectory is determined. The forced delay of the correction may lead to a larger required fuel reserve than that required with another method of correction, which allows for the possibility of carrying out the correction at a moment of time when the motion trajectory still is not known with the given precision.

In [3-5] the correction problem was solved using a unified model. It was assumed that the pulses are applied at specific moments of time and their magnitudes linearly depend on the results of the trajectory measurements which are obtained up to the moment each pulse is applied. The correction moments and the form of the linear dependence of the pulse on the measurement results are determined for which the average fuel consumption is a minimum and the /358 dispersions of the final parameters are limited by given quantities. With this formulation of the problem, only the magnitude of the correcting pulse is determined by the measurement results. The moments for conducting the corrections do not depend on the measurements.

One may investigate strategies in which both the magnitude and the moment of application of each pulse depend on the measurement results. Let us arbitrarily call such strategies flexible as distinct from inflexible strategies in which the correction moments are fixed beforehand. With the flexible strategy, the first correction is called for sooner if the trajectory measurements show that the injection errors are large. A delay in this case may lead to a fuel consumption which exceeds the available reserve. This excess is explained by the reduced effectiveness of the correcting pulse and large

deviations of the actual trajectory from the rated trajectory. If the trajectory measurements show that the injection errors are insignificant, then the corrections may be delayed and can be performed when the trajectory is determined more precisely. Each correction act in the flexible strategy consists of two steps. In the first step, the question is decided as to whether the time for carrying out a given correction has arrived. If this time has not arrived, then further measurements are conducted. If the time has arrived, then the second step begins — that is, the step of making a decision as to the magnitude of the correcting pulse. The decisions in each correction act are made on the basis of the actual trajectory measurements, the fuel remainder which is available at the given moment, and the number of corrections which will be carried out in the future.

Considering what has been discussed, in [3-5] only a small subclass of possible control methods was investigated. The minimum average amount of fuel in this subclass, generally speaking, is larger than in the class of all possible strategies. The optimum criterion itself — that is, the minimum average fuel consumption — is not straightforward, since in the correction problem one is interested not in the average consumption, but in the fuel reserve which would guarantee the reliable fulfillment of the flight assignment.

In this study, the problem of constructing optimum flexible strategies with one or two corrections is investigated. The criterion for being optimum is the maximum probability of impacting in a given region with finite parameters with a limited fuel reserve.

FORMULATION OF THE PROBLEM

A linear model serves as the basis for the investigation. It is assumed that the spacecraft motion in the vicinity of the rated trajectory is described by the system of differential equations

$$\dot{\bar{h}}(t) = Q(t)\bar{h}(t) + Q_1(t)\bar{u}(t), \qquad (1)$$

where $\bar{h}$ (t) is a six-dimensional vector of the phase deviations of the actual trajectory from the rated trajectory; $\bar{u}$ (t) is the control. The initial deviations $\bar{h}$ (0) = $\bar{h}^0$ are called injection errors. Pulse controls which do not have execution errors are investigated. There are a finite number of control pulses in the time interval [0, T], where T is the moment at which the planned motion ends. The control pulse $(\bar{0},\bar{q})$ designates the instantaneous change at the moment τ of the spacecraft velocity without a change of its coordinates. The velocity $\bar{q}$ we call the correcting pulse. A fuel consumption $|\bar{q}|$ is connected with each pulse. We will assume that the fuel reserve for carrying out the corrections is limited and equals W_0.

Let us designate by H (t) the fundamental solution matrix of a homogenous system corresponding to (1). Then the solution of system (1), with consideration of the pulse nature of the control, may be written in the form /359

$$\bar{h}(t) = H(t)\,H^{-1}(0)\,\bar{h}^0 + \sum_{j=1}^{m} H(t)\,H^{-1}(\tau_j)\,(\bar{0},\bar{q}), \qquad (2)$$

where m is the number of correcting pulses in the time interval [0,t]; τ_j are the moments of their application.

Because of the injection errors and because of the control, the actual motion parameters at the time T differ from the rated parameters by an amount $\bar{h}$ (T) which is calculated on the basis of (2). Frequently it is not the deviations themselves $\bar{h}$ (T) which are important to the observer, but their linear combinations which are called correctable parameters or misses. The case of a one-dimensional correctable parameter space will be investigated. For this case, the miss δ at the moment T may be represented in the form

$$\delta = \bar{K}H(T)\,H^{-1}(0)\,\bar{h}^0 + \bar{K}\sum_{j=1}^{n} H(T)\,H^{-1}(\tau_j)\,(\bar{0},\bar{q}_j) = \bar{k}\bar{h}^0 + \sum_{j=1}^{n} \bar{\theta}(\tau_j)\,\bar{q}_j, \qquad (3)$$

where n is the number of pulses in the interval [0, T]. The resulting miss is composed of two terms: the miss due to the injection errors and the shift in the correctable parameter space due to the correcting pulses.

In the correctable parameter space, a domain of permissible misses

$A = (-\delta_0, \delta_0)$ is selected. It is assumed that the correction has been successfully carried out if, at the moment T, the following event occurs

$$\{\delta \in A\}. \tag{4}$$

The observer conducts trajectory measurements for the purpose of obtaining information on the motion. The composition, number and moments of time of the measurements (the latter are designated by t_1 t_2, . . . t_N) are determined beforehand by a prescribed schedule. Let us assume that the deviations of certain quantities from the rated values are measured, that the measured quantities linearly depend on the deviations $\bar{h}$ (t) and that the measurements are conducted with errors. The quantity which is measured at the moment t_i we will designate as

$$\alpha_i = \bar{L}(t_i)\ \bar{h}(t_i).$$

The measurement result is

$$x_i = \alpha_i + \Delta_i, \tag{5}$$

where Δ_i is the error of the individual measurement. Taking into account (2) and (5), the measurement result is written in the form

$$x_i = \bar{L}(t_i)\left[\sum_j H(t_i) H^{-1}(\tau_j)(\bar{0}, \bar{q}_j) + H(t_i) H^{-1}(0)\bar{h}^0\right] + \Delta i = \sum_j \bar{s}^i(\tau_j) \times \\ \times \bar{q}_j + \bar{l}_i \bar{h}^0 + \Delta_i, \tag{6}$$

where τ_j is the moment of the j-th correction in the interval $[0, t_i]$; $\bar{q}_j$ is the correcting pulse at the moment τ_j; $\bar{s}^i(\tau_j)$ is the vector characterizing the effect of the correcting pulse at a moment $t<t_i$ on the value of the quantity being measured at the moment t_i.

Every set ω of injection errors and measurement errors $(\bar{h}^\circ, \Delta_1, \ldots, \Delta_N)$ we will call an elementary result. Let us assume that in the elementary result space Ω the Gaussian probability P is given.

The space $X = \{x\}$, $x = (x_1, \ldots, x_N)$ we call the space of trajectory measurement selection or simply the sample space. On the basis of the sample, the observer designates the time of the corrections and the magnitudes of the /360

correcting pulses.

Cases of one-time and two-time corrections are investigated in the paper. Let us give a definition of the strategy for each of them. The strategy in a one-time correction is the rule $[\tau(x), \bar{q}(x)]$, which makes each set x of observations agree with the correction time τ and the correcting pulse magnitude $\bar{q}$. (The values of the correction moment lie in the time interval [0, T]; the power limitation $|\bar{q}| \leqslant W_0$ is imposed on the correcting pulse magnitude. The strategy must be established in a definite form. Let us assume a certain strategy $[\tau(x), \bar{q}(x)]$ is given. We use B_i to designate the sample set $\{x : t_i \leqslant \tau(x) < t_{i+1}\}$. For such samples, the strategy designates the correction in the interval $[t_i, t_{i+1})$. The question as to whether the sample x belongs to the set B_i is decided only on the basis of its part $\bar{x}_i = (x_1, \ldots, x_i)$. All samples having the origin $\bar{x}_i$ which is common with the sample $x \in B_i$ belong to B_i. Consequently, the sets B_i must be cylindrical with respect to the last N - i coordinates. The strategy is an unequivocal rule which enables one to designate the correction on the basis of x. It follows from this that the sets B_i do not intersect, and in sum they comprise all X. In sets B_i, the strategy gives the correction time functions $t_i \leqslant \tau(\bar{x}_i) < t_{i+1}$ and the correcting pulse magnitudes $\bar{q}(\bar{x}_i)$.

The strategy in the two-time correction is the rule $[\tau_1(x), \bar{q}_1(x), \tau_2(x), \bar{q}_2(x)]$, which enables one to designate, for each set x of observations, the moment and the magnitude of the pulse, first for the first correction and then for the second correction. For such a strategy, one can distinguish cylindrical sets $B_i = \{x : t_i \leqslant \tau_1(x) < t_{i+1}\}$, $i = 1, 2, \ldots, N-1$ with functions for the moment of application and magnitude of the first correction pulse $\tau_1(\bar{x}_i), q_1(\bar{x}_i)$. The sets $\{B_i\}$ will produce a partitioning of X. Each of the sets B_i in turn is divided into the cylindrical sets $B_{ij} = B_i \cap \{x : t_j \leqslant \tau_2(x) < t_{j+1}\}$, $j = i, \ldots N$, in which the functions $\tau_2(\bar{x}_j, i), q_2(\bar{x}_j, i)$ for the second correcting pulse are given.

To each strategy with one or two corrections which is designated hereafter by ν, there corresponds a specific probability of the event (4)

$$p\{\delta \in A\}_\nu = p_\nu.$$

If $\chi_A(\delta)$ designates the characteristic function of interval A, the probability of the correction success may be written in the form

$$p_\nu = \int \chi_A [\delta_\nu(\omega)]\, dP(\omega), \qquad (7)$$

where $\delta_\nu(\omega)$ is a function of the miss in the correctable parameter space. This function depends on the strategy ν and has the form [see relationship (3)] $\delta_\nu(\omega) = \overline{k h^0} + \xi_\nu(\omega)$, where $\xi_\nu(\omega)$ is the shift in the correctable parameter space which is obtained from an application of strategy ν with the result ω. If, for example, ν is the strategy for a one-time correction and ω is an elementary result which leads to a sample from B_i, then $\xi_\nu(\omega) = \bar{\theta}[\tau(x)] \times |q(x_i)|$, where $x_i = (x_1, \ldots, x_i)$ and $x_k = \overline{l_k h^0} + \Delta k$, $k = 1, 2, \ldots, i$.

The strategy ν^* belonging to a given class will be called the optimum strategy in this class if for any other strategy ν of this class $p_\nu \leqslant p_{\nu^*}$.

The problem consists of constructing the optimum strategy in the classes E_1 and E_2 of strategies with one and two corrections.

COMPLETE SUBCLASSES OF STRATEGIES. EQUIVALENT REPLACEMENT OF THE SAMPLE SPACE.

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Let us separate out from the classes E_1 and E_2 the complete subclasses Π_1 and Π_2 of strategies which have an advantage in the method of assignment. The complete subclass Π_m, m = 1, 2, as is known, is called a subset of strategies of the set E_m such that for every strategy $\nu \in E_m$ and $\nu \in \Pi_m$ one is able to find a strategy $\rho \in \Pi_m$, which is no worse than ν with respect to the probability of success in the corrections.

Let us investigate the influence function $\Theta(t)$ which is the modulus of the vector $\Theta(t)$ in Expression (3). It shows how maximum shift in the miss space may be obtained by a correction at the moment t assuming a single fuel reserve. Let us assume $\Theta(t)$ is continuous. Let us designate by Θ_i the maximum of $\Theta(t)$ in the interval $[t_i, t_{i+1}]$. Let us select the moments of time τ_i which satisfy the following conditions: $t_i \leqslant \tau_i < t_{i+1}$, $\Theta(\tau_i) = \Theta_i$, $\Theta_i > \Theta(t)$ with $t > \tau_i$. There must be no more than one such moment in each interval $[t_i, t_{i+1}]$, and if the

influence function is not monotonic such moments may be absent in certain intervals.

By definition, those strategies and only those strategies from E_m belong to the subclass Π_m which allow a correction only at the moment τ_i in the direction $\bar{\theta}(\tau_i)$.

lemma. For every strategy $\nu \in E_m$ there is a strategy $\rho \in \Pi_m$, such that $p_\nu = p_\rho$. The fuel consumption function for ρ does not exceed the consumption function for ν.

The lemma establishes the completeness of the subclass Π_m. Henceforth we will limit ourselves to strategies of the subclass Π_m, since the assignment of such strategies is simpler than the assignment of arbitrary strategies $\nu \in \Pi_m$. It requires only that one indicate the partitioning of the sample space and the correcting pulse functions along fixed directions $\bar{\theta}(\tau_i)$. When assigning a strategy from the subclass Π_m, it is not necessary to indicate the correction time functions — they are determined at once by the partitioning.

The partitioning of the sample space together with the set of functions of the pulse moment and magnitude is the method of assigning the strategy as a function in the elementary result space. It is convenient to assign the investigated subclasses Π_1 and Π_2 as strategy classes in Ω in the Gaussian case by means of another sample space $Z = \{(z_1, \ldots, z_N)\} = \{z\}$, whose coordinates are the differences of the miss predictions during succesive measurements under the conditions of no control. A miss prediction a_i^0 at the moment t_i with no control conditions means the mathematical expectation of a miss $\bar{k}\bar{h}^0$ for known $x_1, \ldots, x_i$, if the control equals zero in the interval $[0, t_i]$. But if the interval $[0, t_i]$ the control does not equal zero, then in determining the prediction without considering the control one should replace the quantities $x_1, \ldots, x_i$ by $y_1, \ldots, y_i$, where $y_j = \bar{l}_j\bar{h}^0 + \Delta_j = x_j - \sum_k \bar{S}^j(\tau_k)\bar{q}_k$. Thus, $z_i = M(\bar{k}\bar{h}^0 \mid y_1, \ldots, y_i) - M(\bar{k}\bar{h}^0 \mid y_1, \ldots, y_{i-1})$. It is easy to show that the investigated subclasses Π_1 and Π_2 may be assigned as strategy classes in Ω by means of Z.

The advantage of introducing Z is explained by the fact that its coordinates are stochastically independent and the precision characteristics for the measurement system are simply expressed in terms of them. In addition to a_i°, let us introduce the concept of the prediction a_i taking into account the control at the moment t_i: $a_i = M(\delta)\, x_1, \ldots, x_i)$, where δ is the miss produced as a result of the injection errors $\bar{h}^\circ$ and as a result of the control in the interval [0, t]. The predictions a_i and a_i° are connected by the obvious relationship $a_i = a_i^0 + \xi_i$. The terms ξ_i are the shift in the space of correctable parameters obtained as a result of control in the interval [0, t]. The dispersion $M(\delta - a_i)^2 = \sigma_i^2$ of the deviation of the miss from its prediction at the moment t_i and the dispersion $M(a_j^0 - a_i^0)^2$ of the random component of the prediction displacement a_j at the moment t_j from its value as the moment t_i, $i < j$ are the precision characteristics of the measurement system. It may be shown that $\sigma_i^2 = \sigma_{i-1}^2 - Mz_i^2$, where $\sigma_0^2 = M(\delta^2)$ is the priori dispersion of the miss, and $M(a_j^0 - a_i^0)^2 = \sum_{i+1}^{j} Mz_k^2$. /362

ONE-TIME CORRECTION

Let us assume that moments of time τ_i which are possible for making the correction exist in each interval $[t_i, t_{i+1}]$. The generalization to the case of a smaller number of possible correction moments is fairly simple. Each strategy $\nu \in \Pi_1$ corresponds to a partitioning of Z into cylindrical sets B_k, k = 1, ..., N. For the samples in B_k, the correction is conducted at the moment τ_k in the direction $\bar{\theta}(\tau_k)$. The function for the correcting pulse given in B_k is represented in the form

$$\bar{q}(\bar{z}_k) = \frac{\bar{\theta}_k}{\theta_k} q(\bar{z}_k),$$

where $q(\bar{z}_k)$ determines the magnitude and sign of the pulse along $\bar{\theta}_k$.

Suppose $B_k^{(\omega)}$ is the preimage of B_k in the elementary result space. On the basis of the additivity of the probability and (7), the probability of the correction success is represented in the form

$$p_\nu = \sum_{k=1}^{N} \int_{B_k^\omega} \chi_A [\delta_\nu(\omega)]\, dP(\omega).$$

By using the concept of conditional probability, let us write p_ν as

$$\sum_{k=1}^{N} \int_{b_k} \left\{ \int \frac{1}{\sqrt{2\pi}\,\sigma_k} \exp - \frac{1}{2\sigma_k^2} \left[\delta - \sum_1^k z_i - q(\bar{z}_k)\,\theta_k \right]^2 d\delta \right\} dP(\bar{z}_k),$$

where b_k is the base of the cylinder B_k. In this expression, the conditional probability of the correction success with known $\bar{z}_k$ and $q(\bar{z}_k)$ is in the brackets.

As the function for the correcting pulse at the moment τ_k, let us investigate

$$q^*(\bar{z}_k) = \begin{cases} -\sum_1^k z_i \\ \frac{1}{\theta_k} \text{ with } \left| \sum_1^k z_i \right| \leqslant W_0 \theta_k, \\ -W_0 \operatorname{sign}\left(\sum_1^k z_i \right) \text{with } \left| \sum_i^k z_i \right| > W_0 \theta_k. \end{cases} \tag{8}$$

It may be seen that this function maximizes the conditional success probability with each $\bar{z}_k$. Therefore, if we set

$$\psi(\bar{z}_k) = \int_A \frac{1}{\sqrt{2\pi}\sigma_k} \exp - \frac{1}{2\sigma_k^2} \left[\delta - \sum_1^k z_i - q(\bar{z}_k)\,\theta_k \right] d\delta,$$

$$p(B_k) = \int_{b_k} \psi(\bar{z}_k)\, dP(\bar{z}_k),$$

then for any function $q(\bar{z}_k) \neq q^*(\bar{z}_k)$

$$p(B_k) \geqslant \int_{b_k} \int_A \frac{1}{2\pi\sigma_k} \exp - \frac{1}{2\sigma_k^2} \left[\delta - \sum_1^k z_i - q(\bar{z}_k)\,\theta_k \right]^2 d\delta\, dP(\bar{z}_k). \tag{9}$$

It follows from (9) that with a fixed partitioning of $\{B_k\}$ the functions $q^*(\bar{z}_k)$ ensure the maximum probability of the correction success. It is natur-/363 al to call them the optimum functions for the correcting pulse.

The optimum partitioning of space Z is done by the method of dynamic

programming. By means of the recurrence formulas

$$r(\bar{z}_N) = \psi(\bar{z}_N),\ r(\bar{z}_{k+1}) = \max\left[\psi(\bar{z}_{k+1}),\ \int r(\bar{z}_{k+r})\, dP(z_{k+2})\right] \tag{10}$$

the sets $C_k = \{z: \psi(\bar{z}_k) > \int r(\bar{z}_{k+1})\, dP(z_{k+1})\}$ are introduced. The optimum partitioning of Z is written in terms of them in the form

$$C_1, \bar{C}_1 \cap C_2, \ldots, \overline{(C_1 \cup \ldots \cup C_{N-1})}.$$

In the investigated Gaussian case, the optimum functions for the pulse and the functions $\psi(\bar{z}_k)$ with a fixed fuel reserve depend only on the miss prediction $\sum_1^k z_i$. The recurrence formulas enable one to conclude that the functions $r(\bar{z}_k)$ depend on the prediction. Let us designate $\psi(\bar{z}_k)$ and $r(\bar{z}_k)$ by $\lambda_k(a)$ and $\mu_k(a)$, where $a = \sum_1^k z_i$.

Let us introduce on the number axis of the set

$$D_k = \left\{a: \lambda_k(a) > \int \mu_{k+1}(a + z_{k+1})\, dP(z_{k+1})\right\},$$
$$k = 1, \ldots, N-1,\ \mu_N(a) = \lambda_N(a). \tag{11}$$

It follows from the formula for C_k and relationships (10) that those samples from Z belong to the set C_k whose sum of the first coordinates is the number from Z. The method for optimally arriving at the solution consists of the following:

1) the correction begins at that moment τ_k when the first-time prediction falls in the set D_k;

2) the magnitude of the correcting pulse is selected in accordance with (8).

In the general case, it is necessary to calculate the domains of D_k using an electronic computer. A case of almost precise measurements exists which permits an approximate analytical investigation. This case is of interest, first of all because of the fact that in practice the measurements may be considered almost precise; after the first measurements, the uncertainty in

the miss is greatly reduced in comparison with the a priori uncertainty. Formally, the condition of almost precise measurements means that the dispersion of the deviation of the miss from the prediction after the first measurement is a great deal smaller than the a priori dispersion:

$$\sigma_1^2 = \sigma_0^2 - Mz_1^2 \ll \sigma_0^2.$$

This inequality, rewritten in the form $\sum_2^N Mz_i^2 \ll \sigma_0^2$, also means that the prediction obtained with the first measurement is only slightly displace in the scale σ_o with subsequent measurements.

Let us note that the random displacement of the prediction which is small in the scale σ_o may be large in the scale of the interval A of permissible misses. Let us write the expression for the conditional probability of the correction success if the correction is conducted at τ_k:

$$\lambda_k(a) = \begin{cases} \Phi(\delta_0/\sigma_k) - \Phi(-\delta_0/\sigma_k) \text{ for } |a| \leqslant W_0\theta_k, \\ \Phi[(\delta_0 - |a| + W_0\theta_k)\sigma_k] - \Phi[(-\delta_0 - |a| + W_0\theta_k)/\sigma_k] \\ \text{for } |a| > W_0\theta_k. \end{cases}$$

The graph of the function $\lambda_k(a)$ has almost a step form. In view of the fact that the dispersion of the prediction a at τ_k which approximately equals σ_o^2 is a great deal larger than σ_k^2 when $|a| > W_0\theta_k$, the function $\lambda_k(a)$ rapidly decreases to zero. The domain in which this decrease takes place has the order of $3\sigma_k \ll 3\sigma_0$. With an increase of the number k, the height of the step increases — that is, σ_k decreases — and the width of the step decreases since θ_k decreases. /364

In the investigated case, the domains of D_k are easily determined. From (10) and (11) for k = N - 1 it is easy to obtain $D_{N-1} = \{a : a > W_0\theta_N\}$ and $\mu_{N-1}(a) \approx \max[\lambda_{N-1}(a), \lambda_N(a)]$.

For the arbitrary k

$$D_k = \{a : a > W_0\theta_{k+1}\}, \qquad \mu_k(a) = \max[\lambda_k, \ldots, \lambda_N].$$

The difference of $\mu_k(a)$ from $\max[\lambda_k, \ldots, \lambda_N]$ occurs in the domains the size of $3\sigma_i$ which surround the points $W_0\theta_i$, $i = k, \ldots, N$.

The total sizes of these domains are small in comparison with $3\sigma_o$ and therefore the optimum success probability which is calculated on the basis of equation $p_{v*} = \int \mu_1(a)dP(z_1)$, may be approximately calculated as $\int_{\max[\lambda_1, \ldots, \lambda_N]} dP(z_1)$. Its value equals

$$2\sum_1^N \left[\Phi\left(\frac{\delta_0}{\sigma_i}\right) - \Phi\left(-\frac{\delta_0}{\sigma_i}\right)\right]\left[\Phi\left(\frac{W_0\theta_i}{\sqrt{Mz_1^2}}\right) - \Phi\left(\frac{W_0\theta_{i+1}}{\sqrt{Mz_1^2}}\right)\right]. \qquad (12)$$

Let us introduce the concept of a minimum fuel reserve for absolutely precise measurements, for which the actual miss is determined after the first measurement. The confidence interval (with a probability of 0.997) of possible miss value equals $3\sigma_o$. Therefore, the fuel reserve necessary for making a correction in this case equals $3\sigma_0/\theta_1$.

Let us analyze equation (12). Let us assume a measurement system such that $\delta_0/\sigma_N \geqslant 3$. Other systems are not of interest, since they do not ensure a success probability close to one. If the dispersion of the random displacement of the prediction after the first measurement is small in the scale A, then δ_0/σ_1 and δ_0/σ_N differ only slightly from one another. It follows from (12) that in this case the success probability is close to one with the minimum allowable fuel reserve. But if the dispersion of the random displacement of the prediction is large in the scale A, then $\delta_0/\sigma_1 \ll \delta_0/\sigma_N$. In this case the success probability is insignificant for a minimum allowable reserve, and in order to obtain a success probability close to one one must increase the fuel reserve.

TWO-TIME CORRECTION

Let us investigate the strategies from the subcalss Π_2. Let us write the success probability of the correction for $v \in \Pi_2$:

$$P_{\nu} = \sum_{1}^{N-1} \int_{b_k} \left\{ \sum_{l=k+1}^{N} \int_{b_{kl}(\bar{z}_k)} \left[\int_A \frac{1}{\sqrt{2\pi}\,\sigma_l} \exp - \frac{1}{2\sigma_l^2} \left(\delta - \sum_1^l z_i - q(\bar{z}_k)\,\theta_k - \right. \right. \right.$$
$$\left. \left. \left. - q(\bar{z}_{l,k})\,\theta_l \right)^2 d\delta \right] dP(\bar{z}_l/z_k) \right\} dP(\bar{z}_k).$$

Here $b_{kl}(z_k)$ designates the set of samples from b_{kl} for which the first k coordinates equal $z_1, \ldots, z_k$.

Let us designate by U_k the subclass of strategies for which the moment of the first correction with a probability of one equals τ_k. The following theorem is valid.

Theorem. In the subclass U_k of strategies with two corrections and with a fixed moment for the first correction, there exists an optimum strategy ν_k^*.

Let us assume the partitioning corresponding to ν_k^* is $\{B_{kl}^{**}\}$, and the correcting pulse function is $q^*(\bar{z}_k)$ and $q^*(\bar{z}_l/k)$, $l = k+1, \ldots, N$. The strategy ν_k^*, which is the optimum strategy in the entire space Z must be the optimum strategy in any of its subsets. This fact enables any strategy $\nu \in \Pi_2$ to be improved by means of the strategies ν_k^*, $k = 1, \ldots, N-1$ in the sense of the probability of the correction success. Actually, let us assume ν is given by the partitioning of $\{B_k, B_{kl}\}$ and the functions $\{q(\bar{z}_k), q(\bar{z}_l, k)\}$. The strategy ρ which is given by the partitioning of $\{B_k, B_k \cap B_{kl}^{**}\}$ and the pulse functions $\{q^*(\bar{z}_k), q^*(\bar{z}_l, k)\}$ will be no worse than $\nu: p_{\rho} \geqslant p_{\nu}$. This means, when ν_k^* are known, that it makes sense to investigate in the subcalss Π_2 only strategies of the form ρ which differ from one another by the partitioning of $\{B_k\}$. One may find the best partitioning from among all the partitionings of $\{B_k\}$. The procedure for obtaining the optimum partitioning of $\{B_k^*\}$ in a two-time correction is similar to the already described procedure for the optimum partitioning of Z in a one-time correction. Let us set

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$$\Psi(\bar{z}_k) = \sum_{l=k+1}^{N} \int_{b_{kl}^{**}(\bar{z}_k)} \left[\int_A \frac{1}{\sqrt{2\pi}\,\sigma_l} \exp - \frac{1}{2\sigma_l^2} \left(\delta - \sum_1^l z_i - q(z_k)\,\theta_k - \right. \right.$$
$$\left. \left. - q(\bar{z}_{l,k})\,\theta_l \right)^2 d\delta \right] dP(\bar{z}_l/\bar{z}_k).$$

Let us also assume

$$R(\bar{z}_N) = \Psi(\bar{z}_N),\ R(\bar{z}_k) = \max\left[\Psi(\bar{z}_k), \int R(\bar{z}_{k+1})\, dP(z_{k+1})\right],$$
$$C'_k = \left\{z : \Psi(\bar{z}_k) > \int R(\bar{z}_{k+1})\, dP(z_{k+1})\right\}.$$

The sets $\{B^*_k\}$ are written in the form

$$C'_1,\ \overline{C'_1} \cap C'_2,\ \overline{(C'_1 \cup C'_2)} \cap C'_3,\ \ldots,\ \overline{(C'_1 \cup \ldots \cup C'_{N-2})}.$$

It may be shown that among the optimum strategies is the strategy in which the control functions for the pulses $Q_1(a, W)$, $Q_2(a, W)$ depend only on the prediction and the fuel remainder. For such a strategy, depending on the fuel reserve, the sets $D_e''(W)$ and $D_k'(W)$ are introduced on the straight line which, as in the case of a one-time correction, determine the optimum partitioning of $\{B^*_k, B^{**}_{kl}\}$.

The behavior with a two-time correction consists of the following. As soon as the prediction of the miss a falls in the set $D_k'(W_0)$, the first correction is called for in accordance with $Q_1(a, W_0)$. The subsequent measurements are conducted and the second correction $Q_2[a', W_0 - |Q_1(a, W_0)|]$ is called for as soon as the following event occurs

$$a' \in D''_e[W_0 - |Q_1(a, W_0)|],$$

where $a' = a + Q_1(a, W_0) + z_{k+1} + \ldots + z_l$.

The algorithm used in the theorem for constructing the optimum strategy in the subclass U_k requires an infinite number of steps and cannot be realized in practice. For this reason, let us investigate an algorithm for the construction of a ε-optimum strategy in the subclass U_k. From the ε-optimum strategies in the subclass U_k, one may construct the ε_1-optimum strategy in the subclass U_k, where $\varepsilon_1 = \varepsilon(N - 1)$.

In the subclass U_k, let us investigate all the strategies with the same function of the first pulse $q(\bar{z}_k)$. From among such strategies, one may separate out by a known method the strategy with a maximum probability. Let us call the strategy the optimum strategy with a given $q(\bar{z}_k)$ and designate it $\nu(q)$.

Let us assume the set of strategies from the subclass U_k is such that /366 the function of the first pulse $q^{(i)}\left(\sum_1^k z_j, W_0\right)$ for this set forms a sequence which converges everywhere to $Q_1\left(\sum_1^k z_j, W_0\right)$. Such a set of strategies always exists. It is sufficient to select

$$q^{(i)}\left(\sum_1^x z_j, W_0\right) = \sum_{j=-2^i}^{2^i} \frac{j}{2^i} W_0 \chi_{\left[W_0 \frac{j}{2^i} < Q_1\left(\sum_1^k z_j, W_0\right) < W_0 \frac{j+1}{2^i}\right]}(z_k)$$

and to take as the set of strategies $\{\nu(q^{(i)})\}$. One may verify that for such strategies the success probability is connected with the optimum probability in the subclass U_k by the inequality

$$p_{\nu_k^*} - p_{\nu(q^{(i)})} \leqslant C \frac{1}{2^i}, \tag{13}$$

where C is a certain constant. The function $Q_1\left(\sum_1^k z_j, W_0\right)$ is known to us. Consequently, the sequence $q^{(i)}\left(\sum_1^k z_j, W_0\right)$ is unknown. However, we know that $q^{(i)}\left(\sum_1^k z_j, W_0\right)$ takes only the values $W_0 \frac{j}{2^i}$, $j = -2^i, \ldots, 2^i$. For each number i, one may indicate that function $q^{[i]}\left(\sum_1^k z_j, W_0\right)$, taking the values $W_0 \frac{j}{2^i}$, for which

$$p_{\nu(q^{[i]})} \geqslant p_{\nu(q^{(i)})}. \tag{14}$$

The function $q^{[i]}$ is found together with the strategy $\nu(q^{[i]})$. The strategy $\nu(q^{[i]})$ is constructed from the strategies $\nu\left(W_0 \frac{j}{2^i}\right)$, $j = -2^i, \ldots, 2^i$, so that it coincides with $\nu\left(W_0 \frac{j}{2^i}\right)$ on that set from Z where $\nu\left(W_0 \frac{k}{2^i}\right)$ is better than all the remaining $\nu\left(W_0 \frac{i}{2^i}\right)$, $j \neq k$. From (14) and (13), we obtain

$$p_{\nu^*} - p_{\nu(q^{[i]})} \leqslant C \frac{1}{2^i}. \tag{15}$$

Inequality (15) enables one to construct the ε-optimum strategy. By means of the ε-optimum strategy in the subclass U_k, one is able to construct the

ε_1-optimum strategy in the subclass Π_2, where $\varepsilon_1 = (N-1)\varepsilon$, in precisely the same way as the strategy ν^*, was constructed from the optimum strategies ν_k^*.

Calculations show that — if the measurements are not almost precise — then the optimum control function for the first pulse at the moment τ_k is smaller in its absolute value than the function (8), that is, undercorrection occurs. With almost precise measurements, the undercorrection disappears. In this case, one is able to propose many behavior methods which are close to the optimum. The simplest method consists of the procedure that after the first measurement the following pulse is transmitted

$$Q_1(a, W_0) = \begin{cases} -\frac{a}{\theta_1} & \text{with } |a| \leqslant W_0\theta_1, \\ -W_0 \operatorname{sign} a & \text{with } |a| > W_0\theta_1. \end{cases}$$

After N measurements, a pulse is carried out according to equation (8) when $W_o = W$, where W is the fuel remainder. /367

The success probability with such a strategy equals

$$2\left[\Phi\left(\frac{\delta_0}{\sigma_N}\right) - \Phi\left(-\frac{\delta_0}{\sigma_N}\right)\right]\left[\Phi\left(\frac{W_0\theta_1}{\sqrt{Mz_1^2}}\right) - \Phi\left(-\frac{W_0\theta_1}{\sqrt{Mz_1^2}}\right)\right].$$

It is close to one with the minimum allowable fuel reserve $W_o = 3\sigma_0/\theta_1$. The error in the success probability in comparison with the optimum tends toward zero with an increase of the measurement precision.

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